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AN APPLICATION OF GENERALIZED WRIGHT FUNCTION ON CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS

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Abstract: The primary aim of the present article is to determine some sufficient coefficient conditions for normalized generalized Wright functions belonging to certain families of analytic univalent functions in conic regions. We also obtain coefficient conditions for inclusion relations between these subclasses under a convolution operator. Finally, we introduce an integral operator involving with normalized generalized Wright functions and obtain some sufficient coefficient conditions for this integral operator belonging to families of univalent functions in conic regions.

Keywords and Phrases: Analytic functions, univalent functions, uniformly convex and starlike functions, generalized Wright functions.

2020 Mathematics Subject Classification: 30C45.

1. Introduction

Let $\mathcal{A}$ represent the family of function $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \in \mathbb{C}, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ with normalization condition $f(0) = f'(0) - 1 = 0$.

Further, S stand the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) which are also univalent in $\mathbb{U}$.

In 1995, Dixit and Pal [5] introduced the class $R^{\tau}(A, B)$ consisting of functions $f(z)$ of the form (1.1) if it satisfies the condition

$$\left|\frac{f'(z) - 1}{(A - B)\tau - B\left(f'(z) - 1\right)}\right| < 1,$$

where $z \in \mathbb{U}$, $\tau \in \mathbb{C}/\{0\}$, $-1 \le B < A \le 1$.
Goodman ([10, 11]) (see also ([6, 8, 13, 16, 26, 27]) introduced uniformly convex and uniformly starlike functions in the following way

A function $f(z)$ is said to be uniformly convex in $\mathbb{U}$, if $f(z)$ is convex and has the property that for every circular arc η contained in $\mathbb{U}$ with centre ρ also in $\mathbb{U}$, the arc $f(\eta)$ is convex. A necessary and sufficient condition for a function $f(z) \in \mathcal{A}$ to be uniformly convex in $\mathbb{U}$ is that

$$\Re\left\{1 + z\frac{f''(z)}{f'(z)}\right\} \ge \left|\frac{zf''(z)}{f'(z)}\right|$$

where $z \in \mathbb{U}$. The class of all functions satisfying the above condition is denoted by UCV.

Similarly, a function $f(z)$ is said to be uniformly starlike in $\mathbb{U}$, if $f(z)$ is starlike in $\mathbb{U}$ and has the property that for every circular arc η contained in $\mathbb{U}$ with centre ρ also in $\mathbb{U}$, the arc $f(\eta)$ is starlike with respect to $f(\rho)$. A starlike function $f(z)$ of the form (1.1) is said to be uniformly starlike in $\mathbb{U}$, if and only if

$$\Re\left\{\frac{f(z) - f(\rho)}{(z - \rho)f'(z)}\right\} > 0, \qquad z \in \mathbb{U},\ z \ne \rho.$$

The class of all functions satisfying the above condition is denoted by UST.

Kanas and Wisniowska ([14, 15]) generalized the families of uniformly convex functions and uniformly starlike functions in to $k-$ uniformly convex functions and $k-$ uniformly starlike functions, denoted by $k - UCV$ and $k - ST$, respectively, and defined as

A function $f(z)$ of the form (1.1) is said to be $k-$ uniformly convex functions if it satisfies the analytic criteria

$$\Re\left\{1 + z\frac{f''(z)}{f'(z)}\right\} \ge k\left|\frac{zf''(z)}{f'(z)}\right|$$

where $0 \le k < \infty$, $z \in \mathbb{U}$.

Similarly, a function $f(z)$ of the form (1.1) is said to be $k-$ uniformly starlike functions if it satisfies the analytic condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} \ge k\left|\frac{zf'(z)}{f(z)} - 1\right|$$

where $0 \le k < \infty$, $z \in \mathbb{U}$.

Further, these results were generalized by Bharti et al. [4].

A function $f(z)$ of the form (1.1) is said to be in the class $\mathcal{S}_\epsilon^*$ if it satisfy the condition

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \epsilon,$$

where $\epsilon > 0$, $z \in \mathbb{U}$.

Similarly, a function $f(z)$ of the form (1.1) is said to be in the class $\mathcal{C}_\epsilon$ if it satisfy the analytic criteria

$$\left|\frac{zf''(z)}{f'(z)}\right| < \epsilon,$$

where $\epsilon > 0$, $z \in \mathbb{U}$.

The classes $\mathcal{S}_\epsilon^*$ and $\mathcal{C}_\epsilon$ were studied earlier by Gangadharan et al. [8].

In 1933 Wright [29] introduced the following function known as Wright function

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\alpha n + \beta)} \tag{1.2}$$

where $\alpha > -1$, $\beta \in \mathbb{C}$.

Wright [29] also proved that this function is an entire function for $\alpha > -1$. The recent applications of Wright function in univalent function theory is given in the work of Al-Hawary *et al.* [2], Joshi *et al.* [12] and Mustafa and Altintas [21]. Shahed and Salem [7] generalized Wright function $W_{\alpha,\beta}(z)$ into $W_{\alpha,\beta}^{\gamma,\delta}(z)$ which is defined as

$$W_{\alpha,\beta}^{\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n} \frac{z^n}{n!\Gamma(\alpha n + \beta)} \tag{1.3}$$

where $\alpha > -1$, $\gamma, \delta, \beta \in \mathbb{C}$ and $(\gamma)_n$ is a Pochhammer symbol and defined as

$$(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} := \begin{cases} 1, & n = 0 \\ \gamma(\gamma+1) \cdots (\gamma+n-1), & n \in \mathbb{N} \end{cases} \tag{1.4}$$

and symbol Γ is the Gamma function. It is easy to verify that the function $W_{\alpha,\beta}^{\gamma,\delta}(z)$ is an entire function of order $\frac{1}{1+\alpha}$.

Porwal and Magesh [23] (see also [25]) introduced normalized generalized Wright function as

$$\begin{aligned} \mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(z) &= \Gamma(\beta) z W_{\alpha,\delta}^{\gamma,\delta}(z) \\ &= \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n} \frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)} \frac{z^{n+1}}{n!} \\ &= z + \sum_{n=2}^{\infty} \frac{(\gamma)_{n-1}}{(\delta)_{n-1}} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \frac{z^n}{(n-1)!}. \end{aligned} \tag{1.5}$$

Now, we recall the definition of Hadamard product (or Convolution) of two function represented in Taylor series form. For deep study on Hadamard product one may refer to recent work of Mulyava *et al.* [25]. The Hadamard product (or Convolution) of two functions $f(z)$ of the form (1.1) and $g(z)$ of the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.6}$$

is given by the power series

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.7}$$

Now, we introduce the convolution operator

$$\begin{aligned} I(\gamma,\delta,\alpha,\beta) f(z) &= \mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\gamma)_{n-1}}{(\delta)_{n-1}} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \frac{a_n}{(n-1)!} z^n. \end{aligned} \tag{1.8}$$

The special functions play an important role in geometric function theory. Several researchers give some nice applications of hypergeometric function [27], Bessel functions [17], Mittag-leffler function [1, 3] and Wright function [21] in geometric function theory and establishes a co-relation between special functions and univalent functions. Motivated by above mentioned work and results on mapping properties of some special functions on univalent functions ([9, 19, 20, 24, 7]) we obtain some sufficient coefficient conditions of normalized generalized Wright functions belonging to certain classes of univalent functions in conic regions. We also find sufficient conditions for inclusion relations between various subclasses of univalent functions under a convolution operator. Finally, we introduce an integral operator involving with normalized Wright function.

2. Preliminary Results

To prove our main results we shall require the following lemmas.

Lemma 2.1. ([5]) *If* $f \in R^{\tau}(A, B)$ *is of the form* (1.1) *then*

$$|a_n| \leq (A - B)\frac{|\tau|}{n}, \quad n \geq 2.$$

The result is sharp.

Lemma 2.2. [15, 28] *Let* $f(z) \in \mathcal{A}$ *be of the form* (1.1). *If for some* $k(0 \leq k < \infty)$ *the following inequality*

$$\sum_{n=2}^{\infty} \{n + (n-1)k\} |a_n| \leq 1, \tag{2.1}$$

is satisfied then $f \in k - ST$.

Lemma 2.3. [14, 28] *Let* $f(z) \in \mathcal{A}$ *be of the form* (1.1). *If for some* $k(0 \leq k < \infty)$ *the following inequality*

$$\sum_{n=2}^{\infty} n \{n(k+1) - k\} |a_n| \leq 1, \tag{2.2}$$

is satisfied then $f \in k - UCV$.

Lemma 2.4. ([14, 28]) *Let* $f(z) \in \mathcal{A}$ *be of the form* (1.1). *If for some* $k(0 \leq k < \infty)$ *the following inequality*

$$\sum_{n=2}^{\infty} n(n-1) |a_n| \leq \frac{1}{k+2}, \tag{2.3}$$

holds true then $f \in k - UCV$.
The result is sharp.

Lemma 2.5. [8] *Let* $f \in \mathcal{A}$ *be of the form* (1.1). *If for some* $\epsilon > 0$ *the inequality*

$$\sum_{n=2}^{\infty} (\epsilon + n - 1) |a_n| \leq \epsilon, \tag{2.4}$$

is satisfied then $f \in \mathcal{S}^*_\epsilon$.

Lemma 2.6. [8] *Let* $f \in \mathcal{A}$ *be of the form* (1.1). *If for some* $\epsilon > 0$ *the inequality*

$$\sum_{n=2}^{\infty} n(\epsilon + n - 1)\,|a_n| \le \epsilon, \qquad (\epsilon > 0) \tag{2.5}$$

then $f \in \mathcal{C}_\epsilon$.

Lemma 2.7. ([23]) *For all* $\gamma, \alpha, \delta \ge 0$, *and* $\beta > 0$, *we have*

1. $\sum\limits_{n=0}^{\infty} \frac{(\gamma)_{n+1}}{(\delta)_{n+1}} \frac{\Gamma(\beta)}{\Gamma(\alpha(n+1)+\beta)} \frac{1}{(n+1)!} = \mathbb{W}^{\gamma,\,\delta}_{\alpha,\,\beta}(1) - 1;$

2. $\sum\limits_{n=0}^{\infty} \frac{(\gamma)_{n+1}}{(\delta)_{n+1}} \frac{\Gamma(\beta)}{\Gamma(\alpha(n+1)+\beta)} \frac{1}{(n)!} = \left(\mathbb{W}^{\gamma,\,\delta}_{\alpha,\beta}\right)'(1) - \mathbb{W}^{\gamma,\,\delta}_{\alpha,\,\beta}(1);$

3. $\sum\limits_{n=0}^{\infty} \frac{(\gamma)_{n+1}}{(\delta)_{n+1}} \frac{\Gamma(\beta)}{\Gamma(\alpha(n+1)+\beta)} \frac{1}{(n-1)!} = \left(\mathbb{W}^{\gamma,\,\delta}_{\alpha,\,\beta}\right)''(1) - 2\left(\mathbb{W}^{\gamma,\,\delta}_{\alpha,\,\beta}\right)'(1) + 2W^{\gamma,\,\delta}_{\alpha,\,\beta}(1);$

4. $\sum\limits_{n=0}^{\infty} \frac{(\gamma)_{n+1}}{(\delta)_{n+1}} \frac{\Gamma(\beta)}{\Gamma(\alpha(n+1)+\beta)} \frac{1}{(n-2)!} = \left(\mathbb{W}^{\gamma,\,\delta}_{\alpha,\,\beta}\right)'''(1) - 3\left(\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}\right)''(1)$
$+ 6\left(\mathbb{W}^{\gamma,\,\delta}_{\alpha,\,\beta}\right)'(1) - 6\mathbb{W}^{\gamma,\,\delta}_{\alpha,\,\beta}(1)$

3. Main Results

In our first result we obtain a sufficient condition for $\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}(z)$ belong to the class $k - ST$.

Theorem 3.1. *Let* $\gamma, \delta, \alpha \ge 0$ *and* $\beta > 0$, *if for some* $k(0 \le k < \infty)$ *the inequality*

$$(k+1)\left(\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}\right)'(1) - k\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}(1) \le 2 \tag{3.1}$$

is satisfied then $\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}(z) \in k - ST$.

Proof. To prove $\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}(z) \in k - ST$, from Lemma 2.2 it is sufficient to prove that

$$\sum_{n=2}^{\infty} \{(k+1)n - k\} \frac{(\gamma)_{n-1}}{(\delta)_{n-1}} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \frac{1}{(n-1)!} \le 1.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}\{(k+1)n-k\}\,\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{(n-1)!}\\
&=\sum_{n=2}^{\infty}\{(k+1)(n-1)+1\}\,\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{(n-1)!}\\
&=(k+1)\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{(n-2)!}+\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{(n-1)!}\\
&=(k+1)\sum_{n=0}^{\infty}\frac{(\gamma)_{n+1}}{(\delta)_{n+1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n+1)+\beta\right)}\frac{1}{n!}+\sum_{n=0}^{\infty}\frac{(\gamma)_{n+1}}{(\delta)_{n+1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n+1)+\beta\right)}\frac{1}{(n+1)!}\\
&=(k+1)\left\{\left(\mathbb{W}^{\gamma,\,\delta}_{\alpha,\beta}\right)'(1)-\mathbb{W}^{\gamma,\,\delta}_{\alpha,\,\beta}(1)\right\}+\mathbb{W}^{\gamma,\,\delta}_{\alpha,\,\beta}(1)-1\\
&=(k+1)\left(\mathbb{W}^{\gamma,\,\delta}_{\alpha,\beta}\right)'(1)-k\mathbb{W}^{\gamma,\,\delta}_{\alpha,\,\beta}(1)-1\\
&\leq 1,\qquad \text{(by given hypothesis).}
\end{aligned}$$

This complete the proof of Theorem 3.1.

Theorem 3.2. *Let $\gamma, \delta, \alpha \geq 0$ and $\beta > 0$, if for some $k(0 \leq k < \infty)$ the inequality*

$$(k+1)\left(\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}\right)''(1)+\left(\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}\right)'(1)\leq 2 \tag{3.2}$$

is satisfied then $\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}(z) \in k-UCV$.

Proof. The proof of above theorem is much akin to the proof of Theorem 3.1. Therefore we omit the details involved.

Theorem 3.3. *Let $\gamma, \delta, \alpha \geq 0$ and $\beta > 0$, if for some $k(0 \leq k < \infty)$ the inequality*

$$(k+1)\left(\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}\right)''(1)\leq \frac{1}{k+2} \tag{3.3}$$

is satisfied then $\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}(z) \in k-UCV$.

Proof. To prove $\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}(z) \in k-UCV$, from Lemma 2.4 it is sufficient to prove that

$$\sum_{n=2}^{\infty}n(n-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{(n-1)!}\leq\frac{1}{k+2}.$$

Now

$$\sum_{n=2}^{\infty} n(n-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{(n-1)!}$$
$$=\sum_{n=2}^{\infty}\{(n-1)(n-2)+2(n-1)\}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{(n-1)!}$$
$$=\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{(n-3)!}+2\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{(n-2)!}$$
$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{n+1}}{(\delta)_{n+1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n+1)+\beta)}\frac{1}{(n-1)!}+2\sum_{n=0}^{\infty}\frac{(\gamma)_{n+1}}{(\delta)_{n+1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n+1)+\beta)}\frac{1}{n!}$$
$$=\left\{\left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}\right)''(1)-2\left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}\right)'(1)+2\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1)\right\}+2\left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}\right)'(1)-2\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1)$$
$$=\left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}\right)''(1)$$
$$\le\frac{1}{k+2},\qquad\text{(by given hypothesis)}.$$

This complete the proof of Theorem 3.3.

Theorem 3.4. *Let $\gamma,\delta,\alpha\ge 0$ and $\beta>0$, if for some $\epsilon>0$ the inequality*

$$\left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}\right)'(1)+(\epsilon-1)\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1)\le 2\epsilon \tag{3.4}$$

is satisfied then $\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(z)\in\mathcal{S}_{\epsilon}^{*}$.

Proof. To prove $\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(z)\in\mathcal{S}_{\epsilon}^{*}$, from Lemma 2.5 it is sufficient to prove that

$$\sum_{n=2}^{\infty}(\epsilon+n-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{(n-1)!}\le\epsilon.$$

Now

$$\sum_{n=2}^{\infty}(\epsilon+n-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{(n-1)!}$$
$$=\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{(n-2)!}+\epsilon\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{(n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{n+1}}{(\delta)_{n+1}} \frac{\Gamma(\beta)}{\Gamma\left(\alpha(n+1)+\beta\right)} \frac{1}{n!} + \epsilon \sum_{n=0}^{\infty} \frac{(\gamma)_{n+1}}{(\delta)_{n+1}} \frac{\Gamma(\beta)}{\Gamma\left(\alpha(n+1)+\beta\right)} \frac{1}{(n+1)!}$$
$$= \left(\mathbb{W}_{\alpha,\beta}^{\gamma,\,\delta}\right)'(1) - \mathbb{W}_{\alpha,\beta}^{\gamma,\,\delta}(1) + \epsilon\left(\mathbb{W}_{\alpha,\,\beta}^{\gamma,\,\delta}(1) - 1\right)$$
$$\leq \epsilon, \quad \text{(by given hypothesis).}$$

This complete the proof of Theorem 3.4.

Theorem 3.5. *Let $\gamma, \delta, \alpha \geq 0$ and $\beta > 0$, if for some $\epsilon > 0$ the inequality*

$$\left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}\right)''(1) + \epsilon\left\{\left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}\right)'(1) - \mathbb{W}_{\alpha,\,\beta}^{\gamma,\,\delta}(1)\right\} \leq \epsilon \tag{3.5}$$

is satisfied then $\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(z) \in \mathcal{C}_{\epsilon}$.

Proof. The proof of above theorem is much similar to the proof of Theorem 3.4. Therefore we omit the details involved.

4. Inclusion Relations

Lemma 4.1. ([23]) *For all $\alpha \geq 0$ and $\beta > \alpha,\ \gamma,\ \delta > 1$, we have*

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n} \frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)} \frac{1}{(n+1)!} = \left(\frac{\delta-1}{\gamma-1}\right) \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left[\mathbb{W}_{\alpha,\,\beta-\alpha}^{\gamma-1,\,\delta-1}(1) - 1\right].$$

Theorem 4.2. *Let $\alpha \geq 0,\ \gamma, \delta > 1,\ \beta > \alpha,\ f \in R^{\tau}(A, B)$ and if the condition*

$$(A-B)|\tau| \left[(k+1)\left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1) - 1\right) - k\left\{\left(\frac{\delta-1}{\gamma-1}\right) \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\mathbb{W}_{\alpha,\beta-\alpha}^{\gamma-1,\delta-1}(1) - 1\right) - 1\right\}\right] \leq 1 \tag{4.1}$$

is satisfied then $I(\gamma, \delta, \alpha, \beta)f(z) \in k - ST$.

Proof. To prove that $I(\gamma, \delta, \alpha, \beta)f(z) \in k - ST$ from Lemma 2.2 it is sufficient to prove that

$$\sum_{n=2}^{\infty} \{(k+1)n - k\} \frac{(\gamma)_{n-1}}{(\delta)_{n-1}} \frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)} \frac{a_n}{(n-1)!} \leq 1.$$

Since $f \in R^{\tau}(A, B)$, then from Lemma 2.1 we have

$$|a_n| \leq \frac{(A-B)|\tau|}{n}.$$

Now

$$\sum_{n=2}^{\infty}\{(k+1)n-k\}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{a_n}{(n-1)!}$$
$$\leq (A-B)|\tau|\left[\sum_{n=2}^{\infty}\{(k+1)n-k\}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{n!}\right]$$
$$= (A-B)|\tau|\left[(k+1)\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{(n-1)!}\right.$$
$$\left.-k\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{n!}\right]$$
$$= (A-B)|\tau|\left[(k+1)\sum_{n=0}^{\infty}\frac{(\gamma)_{n+1}}{(\delta)_{n+1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n+1)+\beta)}\frac{1}{(n+1)!}-k\sum_{n=1}^{\infty}\frac{(\gamma)_{n}}{(\delta)_{n}}\frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)}\frac{1}{(n+1)!}\right]$$
$$= (A-B)|\tau|\left[(k+1)\left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1)-1\right)-k\left\{\left(\frac{\delta-1}{\gamma-1}\right)\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\mathbb{W}_{\alpha,\beta-\alpha}^{\gamma-1,\delta-1}(1)-1\right)-1\right\}\right]$$
$$\leq 1, \qquad \text{(by given hypothesis).}$$

Thus the proof of Theorem 4.2 is established.

Theorem 4.3. *Let $\gamma,\delta,\alpha\geq 0$, $\beta>0$, $f\in R^{\tau}(A,B)$ and if for some $k(0\leq k<\infty)$ the inequality*

$$(A-B)|\tau|\left[(k+1)\left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}\right)'(1)-k\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1)-1\right]\leq 1 \tag{4.2}$$

is satisfied then $I(\gamma,\delta,\alpha,\beta)f(z)\in k-UCV$.

Proof. To prove that $I(\gamma,\delta,\alpha,\beta)f(z)\in k-UCV$ from Lemma 2.3 it is sufficient to prove that

$$\sum_{n=2}^{\infty}n\{(k+1)n-k\}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{a_n}{(n-1)!}\leq 1.$$

Since $f\in R^{\tau}(A,B)$, then from Lemma 2.1, we have

$$|a_n|\leq\frac{(A-B)|\tau|}{n}.$$

Now

$$\sum_{n=2}^{\infty}n\{(k+1)n-k\}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{a_n}{(n-1)!}$$
$$\leq (A-B)|\tau|\left[\sum_{n=2}^{\infty}\{(k+1)n-k\}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{(n-1)!}\right]$$

$$\leq (A-B)|\tau| \left[\sum_{n=2}^{\infty} \{(k+1)(n-1)+1\} \frac{(\gamma)_{n-1}}{(\delta)_{n-1}} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \frac{1}{(n-1)!} \right]$$
$$= (A-B)|\tau| \left[(k+1) \sum_{n=2}^{\infty} \frac{(\gamma)_{n-1}}{(\delta)_{n-1}} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \frac{1}{(n-2)!} \right.$$
$$\left. + \sum_{n=2}^{\infty} \frac{(\gamma)_{n-1}}{(\delta)_{n-1}} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \frac{1}{(n-1)!} \right]$$
$$= (A-B)|\tau| \left[(k+1) \sum_{n=0}^{\infty} \frac{(\gamma)_{n+1}}{(\delta)_{n+1}} \frac{\Gamma(\beta)}{\Gamma(\alpha(n+1)+\beta)} \frac{1}{n!} \right.$$
$$\left. + \sum_{n=0}^{\infty} \frac{(\gamma)_{n+1}}{(\delta)_{n+1}} \frac{\Gamma(\beta)}{\Gamma(\alpha(n+1)+\beta)} \frac{1}{(n+1)!} \right]$$
$$= (A-B)|\tau| \left[(k+1) \left\{ \left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta} \right)'(1) - \mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1) \right\} \right.$$
$$\left. + \mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1) - 1 \right]$$
$$\leq 1, \qquad \text{(by given hypothesis).}$$

Thus the proof of Theorem 4.3 is established.

Theorem 4.4. *Let* $\gamma, \delta, \alpha \geq 0$, $\beta > 0$, $f \in R^{\tau}(A, B)$ *and if for some* $k(0 \leq k < \infty)$ *the inequality*

$$(A-B)|\tau| \left[\left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta} \right)'(1) - \mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1) \right] \leq \frac{1}{k+2} \tag{4.3}$$

is satisfied then $I(\gamma, \delta, \alpha, \beta)f(z) \in k - UCV$.

Proof. To prove that $I(\gamma, \delta, \alpha, \beta)f(z) \in k - UCV$ from Lemma 2.4 it is sufficient to prove that

$$\sum_{n=2}^{\infty} n(n-1) \frac{(\gamma)_{n-1}}{(\delta)_{n-1}} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \frac{a_n}{(n-1)!} \leq \frac{1}{k+2}.$$

Since $f \in R^{\tau}(A, B)$, then from Lemma 2.1, we have

$$|a_n| \leq \frac{(A-B)|\tau|}{n}.$$

Now

$$\sum_{n=2}^{\infty} n(n-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{a_n}{(n-1)!}$$
$$\leq (A-B)|\tau|\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{(n-2)!}$$
$$= (A-B)|\tau|\sum_{n=0}^{\infty}\frac{(\gamma)_{n+1}}{(\delta)_{n+1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n+1)+\beta\right)}\frac{1}{n!}$$
$$= (A-B)|\tau|\left[\left(\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}\right)'(1)-\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}(1)\right]$$
$$\leq \frac{1}{k+2}, \qquad \text{(by given hypothesis).}$$

Thus the proof of Theorem 4.4 is established.

Theorem 4.5. *Let $\gamma,\delta,\alpha\geq 0$, $\beta>0$, $f\in R^{\tau}(A,B)$ and for some $\epsilon>0$ the inequality*

$$(A-B)|\tau|\left[\left(\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}\right)'(1)+(\epsilon-1)\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}(1)-\epsilon\right]\leq\epsilon \tag{4.4}$$

is satisfied then $I(\gamma,\delta,\alpha,\beta)f(z)\in\mathcal{C}_{\epsilon}$.

Proof. To prove that $I(\gamma,\delta,\alpha,\beta)f(z)\in\mathcal{C}_{\epsilon}$ from Lemma 2.6 it is sufficient to prove that

$$\sum_{n=2}^{\infty} n(n+\epsilon-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{a_n}{(n-1)!}\leq\epsilon.$$

Since $f\in R^{\tau}(A,B)$, then from Lemma 2.1, we have

$$|a_n|\leq\frac{(A-B)|\tau|}{n}.$$

Now

$$\sum_{n=2}^{\infty} n(n+\epsilon-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{a_n}{(n-1)!}$$
$$\leq (A-B)|\tau|\left[\sum_{n=2}^{\infty}(n+\epsilon-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{(n-1)!}\right]$$
$$= (A-B)|\tau|\left[\epsilon\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{(n-1)!}\right.$$

$$+\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{(n-2)!}\Bigg]$$

$$=(A-B)|\tau|\left[\epsilon\sum_{n=0}^{\infty}\frac{(\gamma)_{n+1}}{(\delta)_{n+1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n+1)+\beta\right)}\frac{1}{(n+1)!}\right.$$

$$\left.+\sum_{n=0}^{\infty}\frac{(\gamma)_{n+1}}{(\delta)_{n+1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n+1)+\beta\right)}\frac{1}{n!}\right]$$

$$=(A-B)|\tau|\left[\epsilon\left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1)-1\right)+\left\{\left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}\right)'(1)-\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1)\right\}\right]$$

$$\leq\epsilon,\qquad\text{(by given hypothesis).}$$

Thus the proof of Theorem 4.5 is established.

Theorem 4.6. *Let $\gamma,\delta>1,\ \alpha\geq 0,\ \beta>\alpha,\ f\in R^{\tau}(A,B)$ and if for some $\epsilon>0$ the inequality*

$$(A-B)|\tau|\left[\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1)-1+(\epsilon-1)\left\{\left(\frac{\delta-1}{\gamma-1}\right)\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\mathbb{W}_{\alpha,\beta-\alpha}^{\gamma-1,\delta-1}(1)-1\right)-1\right\}\right]\leq\epsilon \tag{4.5}$$

is satisfied then $I(\gamma,\delta,\alpha,\beta)f(z)\in\mathcal{S}_{\epsilon}^{*}$.

Proof. To prove that $I(\gamma,\delta,\alpha,\beta)f(z)\in\mathcal{S}_{\epsilon}^{*}$ from Lemma 2.5 it is sufficient to prove that

$$\sum_{n=2}^{\infty}(n+\epsilon-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{a_n}{(n-1)!}\leq\epsilon.$$

Since $f\in R^{\tau}(A,B)$, then from Lemma 2.1, we have

$$|a_n|\leq\frac{(A-B)|\tau|}{n}.$$

Now

$$\sum_{n=2}^{\infty}(n+\epsilon-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{a_n}{(n-1)!}$$

$$\leq(A-B)|\tau|\left[\sum_{n=2}^{\infty}(n+\epsilon-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{n!}\right]$$

$$=(A-B)|\tau|\left[\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{(n-1)!}\right.$$

$$+(\epsilon-1)\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{n!}\Bigg]$$
$$=(A-B)|\tau|\left[\sum_{n=0}^{\infty}\frac{(\gamma)_{n+1}}{(\delta)_{n+1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n+1)+\beta\right)}\frac{1}{(n+1)!}\right.$$
$$\left.+(\epsilon-1)\sum_{n=0}^{\infty}\frac{(\gamma)_{n+1}}{(\delta)_{n+1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n+1)+\beta\right)}\frac{1}{(n+2)!}\right]$$
$$=(A-B)|\tau|\left[\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}(1)-1+(\epsilon-1)\left\{\left(\frac{\delta-1}{\gamma-1}\right)\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\mathbb{W}^{\gamma-1,\delta-1}_{\alpha,\beta-\alpha}(1)-1\right)-1\right\}\right]$$
$$\leq\epsilon,\qquad\text{(by given hypothesis).}$$

Thus the proof of Theorem 4.6 is established.

5. Integral operator

In this section we introduce a generalized integral operator associated with normalized generalized Wright function in the following way

$$\Omega^{\gamma,\delta}_{\alpha,\beta}(z)=\int_0^z\frac{\mathbb{W}^{\gamma,\delta}_{\alpha,\beta}(t)}{t}dt.$$
$$=z+\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{z^n}{n!}. \tag{5.1}$$

Theorem 5.1. *Let $\gamma,\delta,\alpha\geq 0$ and $\beta>0$, if for some $k(0\leq k<\infty)$ the inequality (3.1) is satisfied then $\Omega^{\gamma,\delta}_{\alpha,\beta}(z)\in k-UCV$.*

Proof. To prove $\Omega^{\gamma,\delta}_{\alpha,\beta}(z)\in k-UCV$ from Lemma 2.3 it is sufficient to prove that

$$\sum_{n=2}^{\infty}n\left\{(k+1)n-k\right\}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{n!}\leq 1.$$

Now

$$\sum_{n=2}^{\infty}n\left\{(k+1)n-k\right\}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{n!}$$
$$=\sum_{n=2}^{\infty}\left\{(k+1)n-k\right\}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{(n-1)!}.$$

Now, applying the same reasoning as applied in Theorem 3.1, we obtain the required result.

Theorem 5.2. *Let $\alpha \geq 0$, $\gamma, \delta > 1$, $\beta > \alpha$. If for some $k (0 \leq k < \infty)$, the inequality*

$$(k+1)\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1) - k\left[\left(\frac{\delta-1}{\gamma-1}\right)\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\mathbb{W}_{\alpha,\beta-\alpha}^{\gamma-1,\delta-1}(1)-1\right)\right] \leq 2 \tag{5.2}$$

is satisfied then $\Omega_{\alpha,\beta}^{\gamma,\delta}(z) \in k-ST$.

Proof. To prove that $\Omega_{\alpha,\beta}^{\gamma,\delta}(z) \in k-ST$ from Lemma 2.2 it is sufficient to prove that

$$\sum_{n=2}^{\infty}\{(k+1)n-k\}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{n!} \leq 1.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}\{(k+1)n-k\}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{n!}\\
&=(k+1)\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{(n-1)!}-k\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{n!}\\
&=(k+1)\sum_{n=0}^{\infty}\frac{(\gamma)_{n+1}}{(\delta)_{n+1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n+1)+\beta)}\frac{1}{(n+1)!}-k\sum_{n=0}^{\infty}\frac{(\gamma)_{n}}{(\delta)_{n}}\frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)}\frac{1}{(n+1)!}\\
&=(k+1)\left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1)-1\right)-k\left[\left(\frac{\delta-1}{\gamma-1}\right)\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\mathbb{W}_{\alpha,\beta-\alpha}^{\gamma-1,\delta-1}(1)-1\right)-1\right]\\
&\leq 1, \qquad \text{(by given hypothesis).}
\end{aligned}$$

Thus the proof of Theorem 5.2 is established.

Theorem 5.3. *Let $\alpha, \gamma, \delta \geq 0$, $\beta > 0$. If for some $k (0 \leq k < \infty)$, the inequality*

$$\left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}\right)'(1) - \mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1) \leq \frac{1}{k+2} \tag{5.3}$$

is satisfied then $\Omega_{\alpha,\beta}^{\gamma,\delta}(z) \in k-UCV$.

Proof. To prove that $\Omega_{\alpha,\beta}^{\gamma,\delta}(z) \in k-UCV$ from Lemma 2.4 it is sufficient to prove that

$$\sum_{n=2}^{\infty}n(n-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{n!} \leq \frac{1}{k+2}.$$

Now

$$\sum_{n=2}^{\infty} n(n-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{n!}$$
$$=\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{(n-2)!}$$
$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{n+1}}{(\delta)_{n+1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n+1)+\beta)}\frac{1}{n!}$$
$$=\left(\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}\right)'(1)-\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1)$$
$$\leq\frac{1}{k+2},\qquad\text{(by given hypothesis)}.$$

Thus the proof of Theorem 5.3 is established.

Theorem 5.4. *Let $\alpha \geq 0$, $\gamma,\delta > 1$, $\beta > \alpha$. If for some $\epsilon > 0$, the inequality*

$$\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1)+(\epsilon-1)\left[\left(\frac{\delta-1}{\gamma-1}\right)\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\mathbb{W}_{\alpha,\beta-\alpha}^{\gamma-1,\delta-1}(1)-1\right)\right]\leq 2\epsilon \tag{5.4}$$

is satisfied then $\Omega_{\alpha,\beta}^{\gamma,\delta}(z)\in\mathcal{S}_{\epsilon}^{*}$.

Proof. To prove that $\Omega_{\alpha,\beta}^{\gamma,\delta}(z)\in\mathcal{S}_{\epsilon}^{*}$ from Lemma **??** it is sufficient to prove that

$$\sum_{n=2}^{\infty}(n+\epsilon-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{n!}\leq\epsilon.$$

Now

$$\sum_{n=2}^{\infty}(n+\epsilon-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{n!}$$
$$=\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{(n-1)!}+(\epsilon-1)\sum_{n=2}^{\infty}\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\frac{1}{n!}$$
$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{n+1}}{(\delta)_{n+1}}\frac{\Gamma(\beta)}{\Gamma(\alpha(n+1)+\beta)}\frac{1}{(n+1)!}+(\epsilon-1)\sum_{n=1}^{\infty}\frac{(\gamma)_{n}}{(\delta)_{n}}\frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)}\frac{1}{(n+1)!}$$
$$=\mathbb{W}_{\alpha,\beta}^{\gamma,\delta}(1)-1+(\epsilon-1)\left[\left(\frac{\delta-1}{\gamma-1}\right)\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\mathbb{W}_{\alpha,\beta-\alpha}^{\gamma-1,\delta-1}(1)-1\right)-1\right]$$
$$\leq 2\epsilon,\qquad\text{(by given hypothesis)}.$$

Thus the proof of Theorem 5.4 is established.

Theorem 5.5. *Let $\gamma, \delta, \alpha \geq 0$ and $\beta > 0$, if for some $\epsilon > 0$ the inequality (3.4) is satisfied then $\Omega_{\alpha,\beta}^{\gamma,\delta}(z) \in \mathcal{C}_\epsilon$.*

Proof. To prove that $\Omega_{\alpha,\beta}^{\gamma,\delta}(z) \in \mathcal{C}_\epsilon$ from Lemma 2.6 it is sufficient to prove that

$$\sum_{n=2}^{\infty} n(n+\epsilon-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{n!} \leq \epsilon.$$

Now

$$\sum_{n=2}^{\infty} n(n+\epsilon-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{n!}$$
$$\sum_{n=2}^{\infty} (n+\epsilon-1)\frac{(\gamma)_{n-1}}{(\delta)_{n-1}}\frac{\Gamma(\beta)}{\Gamma\left(\alpha(n-1)+\beta\right)}\frac{1}{(n-1)!}.$$

Now, applying the same technique adopted in Theorem 3.4 we obtain the required results.

Thus the proof of Theorem 5.5 is established.

6. Conclusion

In the present investigation, we have studied the sufficient conditions of normalized Wright function belonging to the families of $k-$ starlike functions, $k-$uniformly convex functions, $\mathcal{S}_\epsilon^*$ and $\mathcal{C}_\epsilon$. In fact these subclasses are the generalization of various well-known subclasses of univalent functions and hence have a great importance in the theory of univalent functions. We also determine sufficient conditions for inclusion relations between these subclasses. The integral operator associated with normalized generalized Wright function are also introduced and we determine some sufficient conditions for this integral operator belonging to these subclasses. We hope that our results establishes a link between univalent function theory with special functions and motivates to researchers for obtaining new results for other special functions in future.

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